ON THE GEOMETRICAL MEAN DISTANCES OF RECTANGULAR AREAS AND THE CALCULATION OF SELF-INDUCTANCE.

By Edward B. Rosa.

1. THE FORMULÆ.

The formula for the self-inductance of a multiple layer coil of rectangular section is obtained by integrating an expression for mutual inductance twice over the area of the section of the coil. If $dxdy$ is an infinitesimal element of the section ABCD at P and $dx'dy'$ is a second element at Q, the mutual inductance of the two circles of which these elementary areas are sections is given by the following formula of Maxwell:[1]

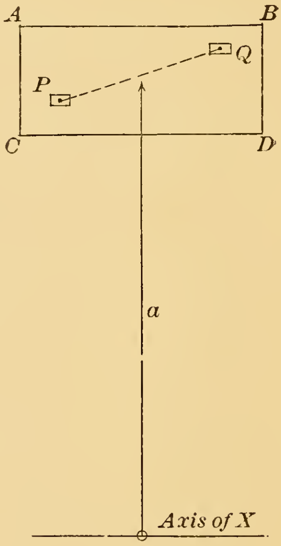

Fig. 1

$$M=4\pi a\left\{\log\frac{8a}{r}\left(1+\frac{y}{2a}+\frac{3x^2+y^2}{16a^2}-\ .\ .\ .\right)-\left(2+\frac{y}{2a}+\frac{x^2-3y^2}{16a^2}-\ .\ .\ .\right)\right\} \quad (1)$$

where a is the radius of the smaller circle, $a+y$ is the radius of the larger circle, x is the distance between their planes and r, the distance between the two infinitesimal sections, is $\sqrt{x^2+y^2}$.

If we integrate equation (1) with respect to $dx'dy'$ over the entire rectangle ABCD and divide by the area of this rectangle, we shall have the mutual inductance of the circle whose section is $dxdy$ and the large ring whose section is ABCD. If we then integrate this expression with respect to $dxdy$ again over the area ABCD (and divide again by the area), we shall have the mutual inductance of the ring whose section is ABCD on itself, which is of course its self-inductance. The current is sup-

[1] Electricity and Magnetism, II, § 705.

posed to be uniformly distributed over the entire cross section of the ring.

If the area be subdivided into n equal squares, it may be taken to represent the section of a circular coil wound with square wire, the latter being insulated by a covering of infinitesimal thickness. The self-inductance of such a coil is n^2 times as great as that of a single ring occupying the same space. Hence by inserting n^2 as a factor in the expression previously derived the formula for the self-inductance of a coil of wire of n turns, mean radius a, axial width b, and radial depth c was derived by Weinstein.[2]

Stefan[3] put the formula into more convenient form for calculation by combining several terms which depend only on the ratio of b to c into two terms, y_1 and y_2, the values of which could be taken from tables prepared by him for the purpose. Stefan's formula is as follows:

$$L=4\pi n^2 a\left\{\log \frac{8a}{\sqrt{b^2+c^2}}\cdot\left(1+\frac{3b^2+c^2}{96a^2}\right)-y_1+\frac{b^2}{16a^2}y_2\right\} \qquad (2)$$

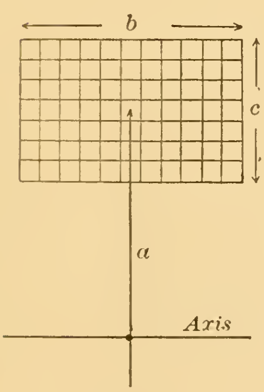
Fig. 2

where b and c are the breadth and depth of the coil as stated above, a is the mean radius, n is the number of turns, and y_1 and y_2 are taken from the table, using $x=\frac{b}{c}$ or $\frac{c}{b}$ as arguments, x being never greater than unity. This table is given in the appendix to this article.

When the section of the coil is square Weinstein's formula is much simplified as follows, where c is the length of one side of the square section:

$$L=4\pi n^2 a\left\{\log \frac{8a}{c}\cdot\left(1+\frac{c^2}{24a^2}\right)+.03657\frac{c^2}{a^2}-1.194913\right\} \qquad (3)$$

Formula (2) may be used where b and c have any ratio; formula (3) applies only where $b=c$. Neither is accurate unless b and c are much less than the radius a.

[2] Wied. Annalen, **21**, pp. 353–354; 1884; Maxwell, Élect. & Mag., 3d ed., II, p. 350.

[3] Wied. Annalen, **22**, p. 107; 1884; and Sitzungsberichte Kais. Akad. der Wiss., Wien, 88 (2), p. 1201; 1883.

2. THE CORRECTION TERMS FOR INSULATED ROUND WIRE.

As stated above, these formulæ apply only to the case of a coil of square wire insulated by a covering of infinitesimal thickness, since the current is supposed to be distributed uniformly over the entire cross section of the coil. This is a condition impossible to realize in practice. When the coil is made up of round wire insulated by a covering of appreciable thickness, the self-inductance is different, as Maxwell pointed out,[4] for three distinct reasons:

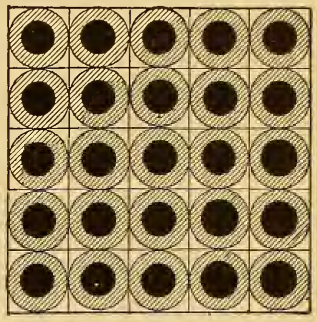

Fig. 3

(1) The self-inductance of each turn of round wire is greater than that of a square wire when the diameter of the round wire is equal to that of the square, since the geometric mean distance R of a square from itself is greater than that of the inscribed circle, and of course (2) it is still greater than that of a smaller circle.

For the square, $\log R_s = \log D + \frac{1}{3}\log 2 + \frac{\pi}{3} - \frac{25}{12}$. For the smaller circle $\log R_c = \log d - \log 2 - \frac{1}{4}$ where D is the side of the square and d is the diameter of the circle. The difference is

$$\log\frac{R_s}{R_c} = \log\frac{D}{d} + \frac{4}{3}\log 2 + \frac{\pi}{3} - \frac{11}{6}$$
$$= \log\frac{D}{d} + 0.1380605 \qquad (4)$$

Hence, the excess of the self-inductance of the coil of round wires due to the greater self-inductance of each of the n turns is

$$\varDelta_1 L = 4\pi n a\left\{\log_e\frac{D}{d} + 0.1380605\right\} \qquad (5)$$

(3) But in addition to this, the mutual inductances of the round wires on each other are different from the mutual inductances of the square wires on one another. Maxwell states[5] that "the induc-

[4] Electricity and Magnetism, II, § 693.

[5] Electricity and Magnetism, II, § 693.

tive action of the eight nearest round wires on the wire A under consideration is less than that of the corresponding eight square wires on the square wire in the middle by 2×0.01971" per unit of length, Fig. 4. "The correction for the wires at greater distance may be neglected," says Maxwell, and hence the total correction may be written, subtracting 0.01971 from 0.13806,

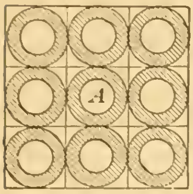

Fig. 4

$$\Delta L = 4\pi na\left\{\log\frac{D}{d} + 0.11835\right\} \qquad (6)$$

This value of ΔL is to be added to L_u computed by Weinstein's or Stefan's formula to give L the corrected self-inductance of the coil.

Stefan's paper giving his modification of Weinstein's formula was published in 1884, and in this paper he stated that Maxwell's value of the absolute term in the correction ΔL (0.11835) was not right,[6] but that it ought to be 0.15494. He did not give his derivation of this constant, but the context suggests that it is obtained by the use of the summation formula for self-inductance, and not as Maxwell had done by means of the geometrical mean distance.

In the third edition of Maxwell the value of the correction due to the difference in the mutual inductances of round wires and of square wires is recomputed by Mr. Chree, using the values of the geometrical mean distances given by Maxwell in his paper "On the Geometrical Mean Distances of Two Figures in a Plane."[7] He obtains substantially the value given previously, namely 0.118425. The correction for reducing from a square to a circle (0.1380605) is right, and Stefan undoubtedly so understood it. The difference between Maxwell and Stefan was as to the correction for the mutual inductances ΔM, where

$$\Delta M = 4\pi anE \qquad (7)$$

According to Maxwell, $E = -.01971$
According to Stefan, $E = +.01688$

That is, the numerical values of the correction are of opposite sign, although of the same order of magnitude numerically. As Stefan did not show how he obtained his value of the correction, and Maxwell

[6] Wied. Annalen, **22**, p. 116; 1884.

[7] Transactions of the Royal Society of Edinburgh, **26**, p. 729; 1872.

did not show how he obtained the values of the geometrical mean distances from which his value is computed, the discrepancy has not been explained. Accordingly, some authors give Maxwell's value and some give Stefan's.

I shall, in what follows, derive the correction first by Maxwell's method, using the geometrical mean distances, and then verify it by means of formulæ of self-inductance. I shall show that Maxwell's values of the geometrical mean distances of neighboring squares are wrong, and hence that his value for E (equation 7) is wrong; that Stefan's value is right as to sign, but only a first approximation as to magnitude; and that the value of this correction term given by Stefan as constant for all cases is not a constant, but is a function of the number of turns in the coil, and the shape of its cross section.

3. THE GEOMETRIC MEAN DISTANCE OF ADJACENT SQUARES.

Maxwell states[7] that the geometrical mean distance of two squares side by side is .99401 times the distance of their centers of gravity, and that the g. m. d. of two squares corner to corner is 1.0011 times the distance of their centers of gravity. He does not, however, give the general expression for these distances.

Gray[8] gives an excellent treatment of the subject of geometrical mean distances, but there are a number of misprints in equations 104, 109, 111, and 113 which it is necessary to correct before using them in numerical computations.[9]

Equation (8) below is Gray's equation (113) corrected, with c in place of b' and d in place of a'.

$$\alpha=\frac{1}{2}\left(d-a\right)$$

$$\beta=\frac{1}{2}\left(d+a\right)$$

[8] Absolute Measurements, Vol. II, Part I, pp. 288–306.

[9] The sign of the first term of equation 104 should be $+$. The signs before p^2 in the coefficients of the log in the first four terms of equation 109 should all be minus; thus $\frac{1}{4}(\beta^2-p^2), -\frac{1}{4}(\alpha^2-p^2), -\frac{1}{4}[(a-\beta)^2-p^2], +\frac{1}{4}[(a-\alpha)^2-p^2]$. Similarly in equation 111 the coefficients of the first two terms should be $\frac{1}{2}(\beta^2-p^2)$ and $-\frac{1}{2}(\alpha^2-p^2)$. In equation 113 the coefficient of β^4 in each of the first four terms should be $\frac{1}{6}$ instead of $\frac{1}{2}$ and the first term should have log $[(p+b+b')^2+\beta^2]$ instead of log $[(p+b+b')^2-\beta^2]$.

p = distance between the rectangles, which are symmetrically placed, as shown in Fig. 5. R is the geometrical mean distance between the rectangles.

$$
\begin{aligned}
abcd \log R = & \frac{1}{4}\left[(p+b+c)^2\left\{\beta^2-\frac{(p+b+c)^2}{6}\right\}-\frac{\beta^4}{6}\right]\log\left((p+b+c)^2+\beta^2\right) \\
& -\frac{1}{4}\left[(p+b)^2\left\{\beta^2-\frac{(p+b)^2}{6}\right\}-\frac{\beta^4}{6}\right]\log\left((p+b)^2+\beta^2\right) \\
& -\frac{1}{4}\left[(p+c)^2\left\{\beta^2-\frac{(p+c)^2}{6}\right\}-\frac{\beta^4}{6}\right]\log\left((p+c)^2+\beta^2\right) \\
& +\frac{1}{4}\left[p^2\left\{\beta^2-\frac{p^2}{6}\right\}-\frac{\beta^4}{6}\right]\log\left(p^2+\beta^2\right) \\
& -(\text{the same series of terms with } \beta \text{ replaced by } a) \\
& +\frac{\beta}{3}(p+b+c)^3\tan^{-1}\frac{\beta}{p+b+c}+\frac{\beta^3}{3}(p+b+c)\tan^{-1}\frac{p+b+c}{\beta} \\
& -\frac{\beta}{3}(p+b)^3\tan^{-1}\frac{\beta}{p+b}-\frac{\beta^3}{3}(p+b)\tan^{-1}\frac{p+b}{\beta} \\
& -\frac{\beta}{3}(p+c)^3\tan^{-1}\frac{\beta}{p+c}-\frac{\beta^3}{3}(p+c)\tan^{-1}\frac{p+c}{\beta} \\
& +\frac{\beta}{3}p^3\tan^{-1}\frac{\beta}{p} \qquad +\frac{\beta^3}{3}p\tan^{-1}\frac{p}{\beta} \\
& -(\text{the same series of terms with } \beta \text{ replaced by } a) \\
& -\frac{(\beta^2-a^2)}{8}\left\{(p+b+c)^2-(p+b)^2-(b+c)^2+p^2\right\}-\frac{11}{6}abcd.
\end{aligned}
\tag{8}
$$

Equation (8*a*) below gives the g. m. d. for two *adjacent* symmetrically placed rectangles. The sides of these rectangles are a, b, and c, d, respectively, as before, the distance p apart being reduced to zero, Fig. 5*a*.

$$a=\frac{1}{2}\left(d-a\right)$$

$$\beta=\frac{1}{2}\left(d+a\right)$$

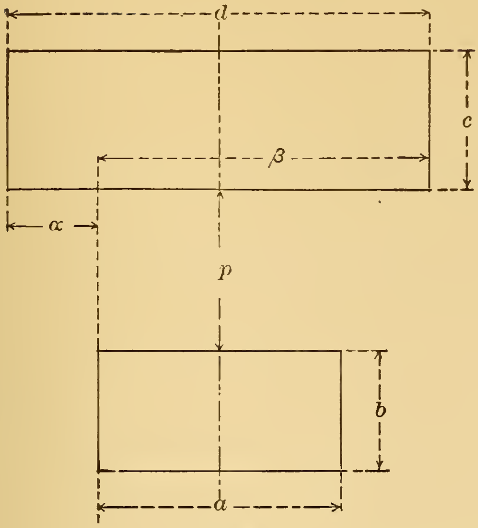

Fig. 5

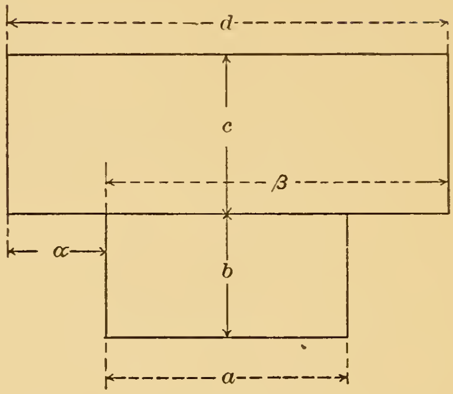

Fig. 5*a*

$$abcd \log R = \frac{1}{4}\left[(b+c)^2\left\{\beta^2 - \frac{(b+c)^2}{6}\right\} - \frac{\beta^4}{6}\right] \log\left\{(b+c)^2+\beta^2\right\}$$

$$-\frac{1}{4}\left[b^2\left\{\beta^2-\frac{b^2}{6}\right\}-\frac{\beta^4}{6}\right]\log(b^2+\beta^2)$$

$$-\frac{1}{4}\left[c^2\left\{\beta^2-\frac{c^2}{6}\right\}-\frac{\beta^4}{6}\right]\log(c^2+\beta^2)-\frac{\beta^4}{24}\log\beta^2$$

$-$(the same series of terms with β replaced by α)

$$+\frac{\beta}{3}(b+c)^3\tan^{-1}\frac{\beta}{b+c}+\frac{\beta^3}{3}(b+c)\tan^{-1}\frac{b+c}{\beta}$$

$$-\frac{\beta b^3}{3}\tan^{-1}\frac{\beta}{b}-\frac{\beta^3 b}{3}\tan^{-1}\frac{b}{\beta}$$

$$-\frac{\beta c^3}{3}\tan^{-1}\frac{\beta}{c}-\frac{\beta^3 c}{3}\tan^{-1}\frac{c}{\beta}$$

$-$(the same series of trigonometrical terms with β replaced by α)

$$-\frac{1}{8}\left(\beta^2-\alpha^2\right)\left\{\left(b+c\right)^2-b^2-c^2\right\}-\frac{11}{6}abcd \qquad (8a)$$

For two adjacent squares of side a, $a=b=c=d$, $\alpha=0$, $\beta=a$, and equation ($8a$) reduces to the following:

$$a^4 \log R_2=\frac{7a^4}{24}\log 5a^2-\frac{a^4}{6}\log 2a^2-\frac{a^4}{6}\log 2a^2-\frac{a^4}{24}\log a^2+\frac{2a^4}{3}\log 4a^2$$

$$-\frac{2a^4}{24}\log a^2+\frac{8a^4}{3}\tan^{-1}\frac{1}{2}+\frac{2a^4}{3}\tan^{-1}2-\frac{4a^4}{3}\tan^{-1}1-\frac{25}{12}a^4$$

$$\therefore \log R_2=\log a+\frac{7}{24}\log 5+\log 2+2\tan^{-1}\frac{1}{2}-\frac{25}{12} \qquad (9)$$

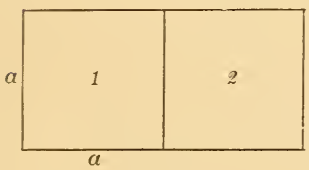

Fig. 6

$$\begin{aligned} \frac{7}{24}\log 5 &= 0.4694194 \\ \log 2 &= 0.6931472 \\ 2\tan^{-1} \tfrac{1}{2} &= 0.9272952 \\ & \quad 2.0898618 \\ -\frac{25}{12} &= 2.0833333 \\ & \quad 0.0065285 = \alpha_2 = \log \frac{R_2}{a} \end{aligned}$$

$$\therefore \log R_2=\log a+.0065285 \qquad (10)$$

$$=\log (1.0065498\, a)$$

$\therefore$ $R_2=1.0065498\ a$, (instead of .99401 a as given by Maxwell)

Therefore, *the g. m. d. of two adjacent squares is 1.0065498 times the distance between their centers of gravity.*

It is to be noticed that formula (9) is an exact one, and any required degree of accuracy can be attained in the value of R by carrying out the values of the four terms to the required number of decimal places. The degree of accuracy above is greater than is ordinarily necessary, but not greater than is desirable for the present purpose.

4. GEOMETRIC MEAN DISTANCE OF OTHER SQUARES IN THE SAME ROW OR COLUMN.

For two squares, as 1 and 3, whose centers are distant $2a$, the geometric mean distance is given by the following expression, obtained from equation (8):

$$\log R_3 = \log a - \frac{7}{4}\log 5 + \frac{27}{4}\log 3 - \frac{11}{3}\log 2 + 8\tan^{-1}\frac{1}{3} - 4\tan^{-1}\frac{1}{2} - \frac{25}{12} \tag{11}$$

Substituting numerical values as before,

$$\begin{aligned} \log R_3 &= \log\ a + .6936576 \\ &= \log\ (2.0010212a) \\ \therefore\ R_3 &= 2.0010212a \qquad\qquad (12) \\ R_3 &= 1.0005106 \times 2a \end{aligned}$$

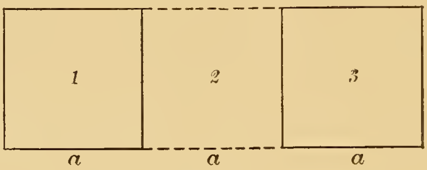

Fig. 7

Therefore, *the g. m. d. of the first and third squares is 1.0005106 times the distance between their centers of gravity.*

Since the g. m. d. between the circles inscribed in these two squares is the distance between their centers, the excess of the log R_3 over log R'_3 (which is log 2) is $0.6936576 - .6931472 = .0005104 = \alpha_3$. Thus α_3 is the naperian logarithm of $1.0005106 = R_3 \div 2a$.

We may obtain a check on the values of α_2 and α_3 by the following method. We first obtain the geometric mean distance of square 1 (Fig. 7) from the rectangle made up of squares 2 and 3, by means of equation (8*a*). The equation will be,

$$\begin{aligned} \log R_s = \log a &+ 4\tan^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{2} - \frac{3}{4}\log 2 - \frac{7}{48}\log 5 \\ &+ \frac{27}{8}\log 3 - \frac{7}{12}\log 10 - \frac{25}{12} \\ &= 0.35009307 + \log a \\ \therefore\ 2\log R_s &= 0.70018606 + 2\log a \end{aligned}$$

The geometric mean distance between the circles inscribed in these squares being the distances of their centers, we have, when $a=1$,

$$2\log R_r = \log 1 + \log 2 = .69314718$$

Thus, $\quad 2\log R_s - 2\log R_r = \alpha_2 + \alpha_3 = .0070389$

Above we found $\quad \alpha_2 = .0065285$

$\qquad\qquad\qquad\qquad \alpha_3 = .0005104$

Sum $= \alpha_2 + \alpha_3 = .0070389$

This exact agreement between the results of two independent calculations assures the correctness of α_2 and α_3.

The g. m. d. of the first and fourth squares we will obtain by the second of the above methods. In equation (8a) substitute $b=a$, $c=3a$, $d=a$, and we find the g. m. d. of the square ABEF from the rectangle BCDE. In this way we find

$$3 \log R_{2.3.4} = 3 \log a + \frac{131}{3} \log 2 - \frac{27}{4} \log 3 + \frac{7}{6} \log 5 - \frac{161}{24} \log 17$$
$$+ 20 \tan^{-1}\frac{1}{4} - 8 \tan^{-1}\frac{1}{3} - \frac{25}{4} \qquad (13)$$
$$= 3 \log a + 1.7989008 \qquad (14)$$

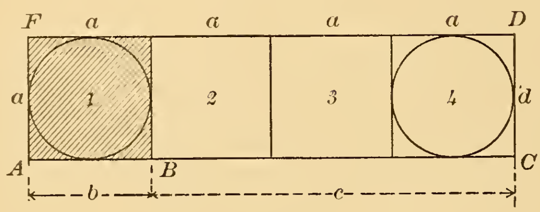

Fig. 8

By the theory of the geometrical mean distance the logarithms of the g. m. d. of the rectangle BCDE from the square is one-third the sum of the logarithms of the g. m. d.'s of each of the squares 2, 3, and 4, which make up the rectangle, from the first square. Therefore, since

$$\log R_2 = \log a + .0065285$$
$$\log R_3 = \log a + .6936576$$
and $\log R_4 = 3 \log R_{2.3.4} - (\log R_2 + \log R_3)$, we have
$$\log R_4 = \log a + 1.0987148$$
$$= \log (3.000307a),$$
$$\therefore R_4 = 1.000102 \times a$$

The g. m. d. of the circle inscribed in square 4 from the circle inscribed in square 1 is $3a$, the distance between their centers; therefore

$$\log R'_4 = \log 3a = \log a + 1.0986123$$
$$\log R_4 - \log R'_4 = .0001025 = a_4$$

Proceeding in the same way for the g. m. d. of squares 5 and 6, with respect to square 1, we find

$$\log R_5 - \log R'_5 = .0000325 = a_5$$
$$\log R_6 - \log R'_6 = .0000136 = a_6$$

In the calculation of the corrections due to the differences in the mutual inductances for round wires from their values for square wires the differences in logarithms of the g. m. d.'s are used, and not the g. m. d.'s themselves. Hence the differences a_2, a_3, etc., are the important quantities to determine, rather than R_1, R_2, etc. All the differences for parallel squares in the same row we have found to be positive, which shows that the mutual inductances of straight round wires at a given distance apart are greater than for square wires at the same distances. The same would be true for the squares in the same column with square number 1. Thus, we have found all the corrections to be applied to the middle wire for the 20 wires in the same row and column for a group of 121 making up a square 11 by 11. These corrections decrease rapidly as we go away from the wire in question, the first correction term being 10 times the sum of the next four.

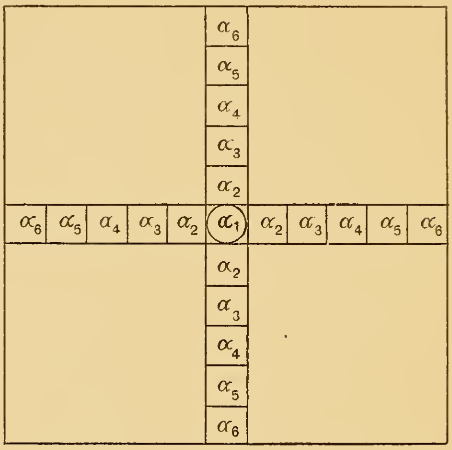

Fig. 9

5. CORRECTIONS FOR THE SECOND ROW.

If we find the g. m. d. of square 1 to the various squares in the second row, we can get the differences in the logarithms of the g. m. d.'s for squares and for circles; that is, the corrections β_1, β_2, β_3,

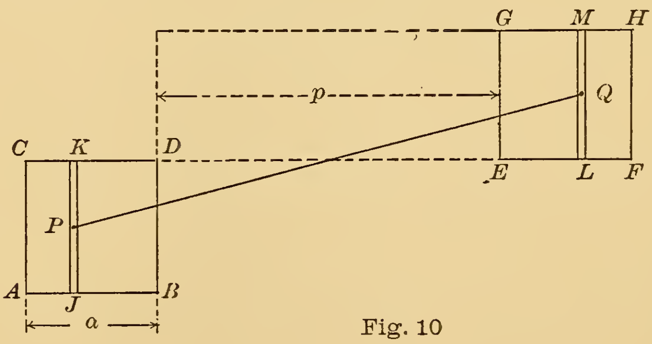

Fig. 10

etc., as we have found a_2, a_3, a_4, etc., for the first row. Taking two squares, as shown in Fig. 10, we express the logarithm of the distance PQ in terms of the coordinates of P and Q. First, we integrate with respect to P along the line JK, and find the g. m. d. from Q to the line JK. Second, integrate with respect to Q along LM, and so find the g. m. d. of LM from JK. Third, integrate with respect to JK along AB and find the g. m. d. of LM from the square ABCD. Fourth, integrate along EF and so find the g. m. d. of one square from the other.

The formula for the g. m. d. of two squares as shown in Fig. 10, their horizontal distance apart being p, determined as above is as follows:

$$\begin{aligned}
\log R_\beta = &\left[\frac{(p+2a)^2}{2a^2}-\frac{(p+2a)^4}{48a^4}-\frac{1}{3}\right]\log\left(4a^2+(p+2a)^2\right)\\
&-\left[\frac{(p+a)^2}{a^2}-\frac{(p+a)^4}{24a^4}-\frac{2}{3}\right]\log\left(4a^2+(p+a)^2\right)\\
&+\left[\frac{p^2}{2a^2}-\frac{p^4}{48a^4}-\frac{1}{3}\right]\log\left(4a^2+p^2\right)\\
&-\left[\frac{(p+2a)^2}{4a^2}-\frac{(p+2a)^4}{24a^4}-\frac{1}{24}\right]\log\left(a^2+(p+2a)^2\right)\\
&+\left[\frac{(p+a)^2}{2a^2}-\frac{(p+a)^4}{12a^4}-\frac{1}{12}\right]\log\left(a^2+(p+a)^2\right)\\
&-\left[\frac{p^2}{4a^2}-\frac{p^4}{24a^4}-\frac{1}{24}\right]\log\left(a^2+p^2\right)\\
&-\frac{(p+2a)^4}{24a^4}\log(p+2a)+\frac{(p+2a)^4}{12a^4}\log(p+a)-\frac{p^4}{24a^4}\log p\\
&+\frac{1}{3}\left[\frac{4p}{a}+8-\frac{(p+2a)^3}{a^3}\right]\tan^{-1}\frac{p+2a}{2a}-\frac{2}{3}\left[\frac{4p}{a}+4-\frac{(p+a)^3}{a^3}\right]\tan^{-1}\frac{p+a}{2a}\\
&-\frac{1}{3}\left[\frac{p}{a}+2-\frac{(p+2a)^3}{a^3}\right]\tan^{-1}\frac{p+2a}{a}+\frac{2}{3}\left[\frac{p}{a}+1-\frac{(p+a)^3}{a^3}\right]\tan^{-1}\frac{p+a}{a}\\
&+\frac{1}{3}\left[\frac{4p}{a}-\frac{p^3}{a^3}\right]\tan^{-1}\frac{p}{2a}-\frac{1}{3}\left[\frac{p}{a}-\frac{p^3}{a^3}\right]\tan^{-1}\frac{p}{a}\\
&-\frac{1}{8}\left[\frac{(p+2a)^2}{a^2}-2\frac{(p+a)^2}{a^2}+\frac{p^2}{a^2}\right]-\frac{11}{6}
\end{aligned}\qquad(17)$$

For two squares corner to corner the distance p *is zero* and equation 17 reduces to the following:

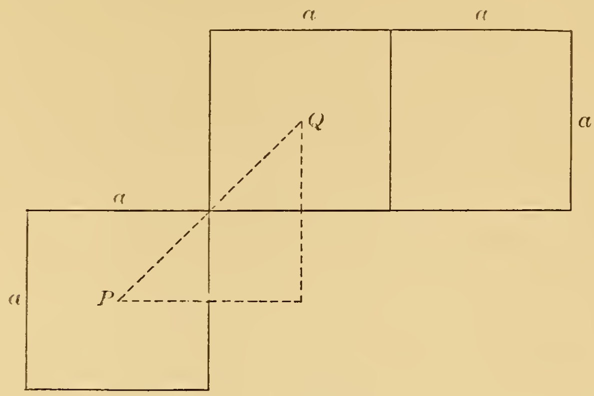

Fig. 11

$$\log R_{\beta_1} = \log a + 3 \log 2 - \frac{7}{12} \log 5 + \pi - 4 \tan^{-1} \frac{1}{2} - \frac{25}{12}$$
$$= \log a + 0.34427164 \qquad (18)$$
$$= \log (1.410962\ a)$$

The distance between centers of these two squares is $a\sqrt{2}$ Therefore,

$$\log R'_{\beta_1} = \log a + \frac{1}{2} \log 2 = \log a + .3465736$$
$$\log R_{\beta_1} - \log R'_{\beta_1} = -.0023019 = \beta_1$$

Also, $R_{\beta_1} = 0.997701 \times a\sqrt{2}$ (instead of $1.0011 \times a\sqrt{2}$ as given by Maxwell). That is, *the g. m. d. of two squares, corner to corner, is 0.997701 times the distance between their centers.*

In a similar manner, putting $p = a$ in equation (17) we find the g. m. d. for the next square in the second row, and also the correction, β_2. Thus,

$$\log R_{\beta_2} = \log a - \frac{17}{3} \log 2 - \frac{27}{8} \log 3 + \frac{91}{48} \log 5 + \frac{119}{48} \log 13$$
$$-\frac{\pi}{2} - 8 \tan^{-1} \frac{1}{3} + 5 \tan^{-1} \frac{1}{2} + 5 \tan^{-1} \frac{2}{3} - \frac{25}{12} \qquad (19)$$
$$= \log a + .8046295$$

$$\log R'_{\beta_2} = \log a + \frac{1}{2} \log 5$$
$$= \log a + .8047190$$
$$\therefore \log R_{\beta_2} - \log R'_{\beta_2} = -.0000895 = \beta_2$$

In a similar manner we find, putting $p=2a$ in equation (17)

$$\log R_{\beta_3}=\log a-\frac{47}{3}\log 2+\frac{27}{4}\log 3-\frac{7}{24}\log 5$$

$$-\frac{119}{24}\log 13+\frac{161}{24}\log 17$$

$$-10\tan^{-1}\frac{2}{3}-20\tan^{-1}\frac{1}{4}+14\tan^{-1}\frac{1}{2}+16\tan^{-1}\frac{1}{3}-\frac{25}{12} \quad (20)$$

$$=\log a+1.1513161$$

$$\log R'_{\beta_3}=\log a+\frac{1}{2}\log 10=\log a+1.1512925$$

$$\therefore \log R_{\beta_3}-\log R'_{\beta_3}=+.00002355=\beta_3$$

It is to be noticed that the correction β_3 is positive, whereas β_1 and β_2 have been negative.

6. CHECK ON VALUES OF β_1, β_2, β_3.

We may obtain a check on the values of β_1, β_2, β_3 just calculated by formula (17), by means of formula (8). The logarithm of the g. m. d. of the square ABCD from the square EFGH (Fig. 12) is equal to the sum of the logarithms of the g. m. d.'s of A for the nine squares making up EFGH. Since a_2 a_3, a_4 are the differences of the logarithms of the corresponding squares and the inscribed circles, we may compute the sum of these differences by means of equation (8a) and compare with the values found. This will give a check on the values of the βs.

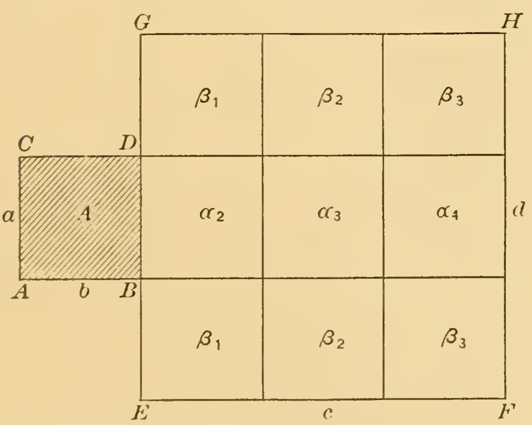

Fig. 12

Substituting in equation (8a) $a=1$, $\beta=2$, $b=1$, $c=3$, $a=1$, $d=3$ we obtain the following equation for the value of the log R for the adjacent squares ABCD and EFGH, Fig. 12:

$$9\log R=9\log a+7\log 2+\frac{77}{24}\log 5-\frac{119}{24}\log 13+\frac{161}{24}\log 17$$

$$+30\tan^{-1}\frac{1}{2}+8\tan^{-1}\frac{1}{3}-20\tan^{-1}\frac{1}{4}-10\tan^{-1}\frac{2}{3}+\pi-\frac{75}{4}$$

$$=9\log a+6.3993349 \quad (21)$$

For the corresponding inscribed circles we have

$$9 \log R' = 9 \log a + 3 \log 2 + \log 3 + 2 \log 5$$
$$= 9 \log a + 6.3969297 \qquad (22)$$

$$\therefore 9 \log R - 9 \log R' = .0024056 = \alpha_2 + \alpha_3 + \alpha_4 + 2(\beta_1 + \beta_2 + \beta_3) \qquad (23)$$

This is the algebraic sum of the αs and βs for the nine squares as given by means of equation (8). We have already checked the values of the αs; if the αs and βs sum up to the above value we shall have also a check on the βs.

Thus

$$\begin{array}{rr} \alpha_2 = & .0065285 \\ \alpha_3 = & .0005104 \\ \alpha_4 = & .0001024 \\ 2\beta_3 = & .0000471 \\ \hline + & .0071884 \end{array}$$

$$2\beta_1 = -.0046038$$
$$2\beta_2 = -.0001790 = -.0047828$$

$$\text{Sum} = +.0024056 = \alpha_2 + \alpha_3 + \alpha_4 + 2(\beta_1 + \beta_2 + \beta_3) \qquad (24)$$

This complete agreement with the value above (23) checks the correctness of the calculations of the βs made from formula (17).

Instead of finding β_4 by equation (17) we may use equation ($8a$),

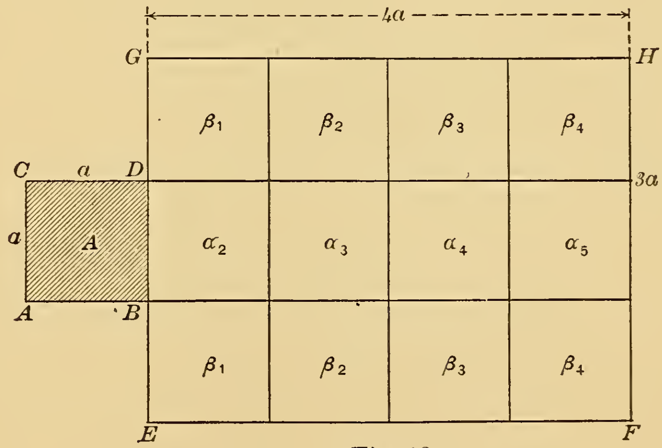

Fig. 13

thus: Find log R for square A with respect to square EFGH, and log R' for circle A with respect to the 12 circles inscribed in the 12 squares of EFGH. The difference gives the sum of the αs and βs.

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Since all are separately known except β_4, the latter then becomes known. Thus, by means of equation ($8a$) we have

$$
\begin{aligned}
12 \log R = & \ 12 \log a + \frac{28}{3}\log 2 - \frac{119}{24}\log 5 + \frac{119}{6}\log 13 \\
& - \frac{161}{24}\log 17 - \frac{41}{24}\log 29 + 20 \tan \frac{1}{4} - 40 \tan^{-1}\frac{1}{5} \\
& + 70 \tan^{-1}\frac{2}{5} - 34 \tan^{-1}\frac{1}{2} + \pi - 25 \\
= & \ 12 \log a + 10.6189076
\end{aligned}
\qquad (25)
$$

For circle A and the 12 circles of EFGH we have

$$
\begin{aligned}
12 \log R' &= 12 \log a + 5 \log 2 + \log 3 + 2 \log 5 + \log 17 \\
&= 12 \log a + 10.6164374
\end{aligned}
$$

$$
\begin{aligned}
\therefore 12 \log R - 12 \log R' &= .0024702 = a_2 + a_3 + a_4 + a_5 + 2(\beta_1 + \beta_2 + \beta_3 + \beta_4) \\
\text{By (23),} \quad &\ \ .0024056 = a_2 + a_3 + a_4 \qquad + 2(\beta_1 + \beta_2 + \beta_3) \\
\therefore &\ \ \overline{.0000646} = a_5 + 2\,\beta_4 \\
\text{But} &\ \ .0000325 = a_5 \\
\therefore &\ \ \overline{.0000321} = 2\,\beta_4 \\
&\ \ .0000160 = \beta_4
\end{aligned}
$$

7. CORRECTION FOR THE FIRST TWO LAYERS OF WIRE.

Considering the square ABCD made up of 25 smaller squares, the geometric mean distances of the middle square A from the rectangle CDEF and from the rectangle FGHI are given by formula ($8a$). If R_{10} is the geometric mean distance of A from the rectangle CDEF (made up of 10 small squares) and R_2 is the g. m. d. of A from the rectangle FGHI (consisting of 2 small squares), we have from formula ($8a$) when $a = 1$,

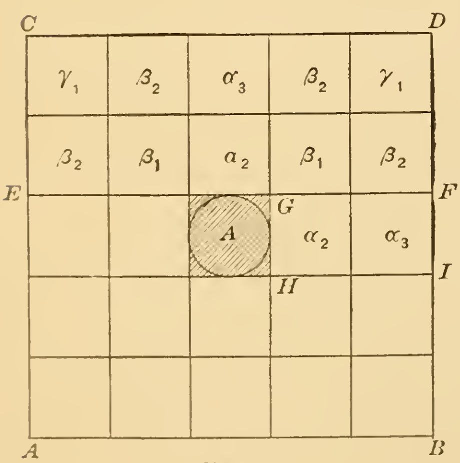

Fig. 14

$$
\begin{aligned}
10 \log R_{10} &= 6.68642698 \\
2 \log R_2 &= 0.70018606
\end{aligned}
\qquad (26)
$$

Adding these quantities and dividing by 12 we get the log of the g. m. d. of A from the whole space outside A inclosed within the larger square, that is, of A from the 24 small squares outside A. Calling this latter g. m. d. R_{24} we have

$$\log R_{24} = .61555109.$$

The g. m. d. of the circle inscribed in A from a circle inscribed in any of the other squares is of course the distance between their centers. The log of the g. m. d. of the circle in A from the 24 circles in the other squares is found from the following expression:

$$24 \log R_{24} = 4 \log 1 + 12 \log 2 + 4 \log 5$$
$$= 14.75551782 \tag{27}$$
$$\therefore \log R'_{24} = .61481324.$$

The mean correction for the mutual inductance of tne 24 wires on the middle wire is the difference of these logarithms. Thus

$$\log R_{24} - \log R'_{24} = .00073785.$$

The total correction for the 24 wires making up the two layers surrounding wire A is 24 times as great, namely,

$$\Delta M_2 = .017708$$

The correction for the eight wires constituting the first layer surrounding A is 4 $(\alpha_2 + \beta_1) = 4(.0065285 - .0023019)$. Thus

$$\Delta M_1 = .016906$$

Thus the difference, representing the correction due to the 16 wires in the second layer is about 5 per cent of the effect of the first 8 wires.

8. THE CORRECTIONS, γ_1, γ_2, γ_3, δ_1.

We have found the corrections for all the wires separately except the corner ones, the correction for which is γ_1. The sum of the corrections for the six wires of one quadrant is one-fourth of ΔM_2, or .00442707. This is equal to

$$\alpha_2 + \alpha_3 + \beta_1 + 2\beta_2 + \gamma_1$$

$$\begin{aligned} \alpha_2 + \alpha_3 &= .0070389 \\ \beta_1 + 2\beta_2 &= -.0024810 \\ \hline \therefore\ \alpha_2 + \alpha_3 + \beta_1 + 2\beta_2 &= .0045579 \\ \text{Sum of } 6 &= .0044271 \\ \hline \therefore\ \gamma_1 &= -.0001308 \end{aligned}$$

In a similar manner, calculating the log of the g. m. d. of square A from the 15 squares making up the rectangle ABCD, and getting the excess over the log for the corresponding inscribed circles we find the sum of the corrections for the 15 squares. All of them having been previously calculated except γ_2, the latter becomes known. Thus,

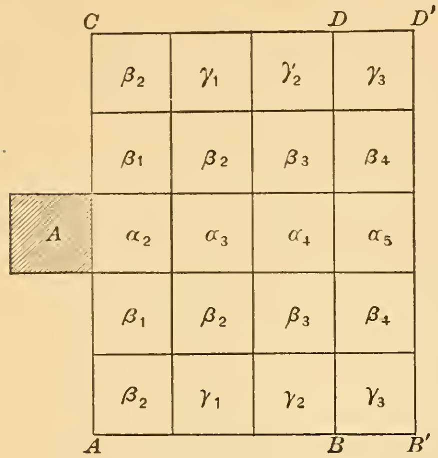

Fig. 15

$$\gamma_2 = -.0000347$$

Instead of doing similarly with the rectangle AB′CD′ to get γ_3, I have computed the g. m. d. of the square A with respect to the rectangle BB′DD′, consisting of 5 squares. The not very simple expression is as follows:

$$\begin{aligned}5 \log R &= 59 \log 2 + 27 \log 3 - \frac{471}{6} \log 5 - \frac{119}{24} \log 13 \\ &+ \frac{161}{6} \log 17 + \frac{41}{24} \log 29 + 80 \tan^{-1}\frac{3}{5} + 64 \tan^{-1}\frac{1}{2} \\ &- 56 \tan^{-1}\frac{3}{4} - 70 \tan^{-1}\frac{2}{5} - 10 \tan^{-1}\frac{2}{3} - \frac{125}{12} \qquad (28) \\ &= 7.2152930\end{aligned}$$

$5 \log R' = 7.2152400$, for the corresponding inscribed circles.

Difference $= 0.0000530 = a_5 + 2\beta_4 + 2\gamma_3$

$$\begin{aligned} a_5 &= .0000325 \\ 2\beta_4 &= .0000321 \\ \because a_5 + 2\beta_4 &= .0000646 \\ \text{Sum of } 5 &= .0000530 \\ \because 2\gamma_3 &= -.0000116 \\ \gamma_3 &= -.0000058 \end{aligned}$$

To get these small corrections accurately by this method of difference requires extending the values of the logarithms and inverse tangents to seven or eight decimal places. The above expression for $5 \log R$ gives that quantity as the difference between large positive and negative quantities (225.24224423 − 218.02695118), and then the total correction for the 5 squares is the very small difference between two large quantities (7.2152930 − 7.2152400). This small difference is then the sum of 5 corrections, of which 3 are known. That these final small differences which are the corrections sought are accurately determined is proved by the exact agreement of their values as checked by different formulæ.

The value of δ_1 is $-.0000256$. Mr. F. W. Grover has kindly calculated for me the values of γ_4, δ_2, δ_3, ϵ_1, ϵ_2, ζ_1, and verified the calculation of all the others. Figure (16) gives the 30 corrections for one quadrant; the other quadrants would of course have the same corrections for squares in corresponding positions. It will be noted that the *a* corrections at the right and left and above and below the square A are positive, whereas the corrections for squares at 45° with the principal axes as β_1, γ_1, δ_1, ϵ_1, ζ_1, are negative. On either side of the 45° lines the corrections are also negative, as β_2, γ_2, γ_3, etc. Nearer the axes the corrections become positive, as β_3, β_4, β_5, γ_4. The correction δ_1 is four times γ_3, although the squares are almost equally distant from A, because γ_3 lies at a lower angle, somewhat off the diagonal.

ζ_1	ε_2	δ_3	γ_4	β_5	α_6	+8.4 β_5	+0.7 γ_4	−4.0 δ_3	−4.5 ϵ_2	−3.1 ζ_1
ε_2	ε_1	δ_2	γ_3	β_4	α_5	+16.0 β_4	−5.8 γ_3	−11.3 δ_2	−8.1 ϵ_1	−4.5 ϵ_2
δ_3	δ_2	δ_1	γ_2	β_3	α_4	+23.6 β_3	−34.8 γ_2	−25.6 δ_1	−11.3 δ_2	−4.0 δ_3
γ_4	γ_3	γ_2	γ_1	β_2	α_3	−89.5 β_2	−130.8 γ_1	−34.8 γ_2	−5.8 γ_3	+0.7 γ_4
β_5	β_4	β_3	β_2	β_1	α_2	−2301.9 β_1	−89.5 β_2	+23.6 β_3	+16.0 β_4	+8.4 β_5
α_6	α_5	α_4	α_3	α_2	A	+6528.5 a_2	+510.4 a_3	+102.5 a_4	+32.5 a_5	+13.6 a_6
β_5	β_4	β_3	β_2	β_1	α_2	β_1	β_2	β_3	β_4	β_5
γ_4	γ_3	γ_2	γ_1	β_2	α_3	β_2	γ_1	γ_2	γ_3	γ_4
δ_3	δ_2	δ_1	γ_2	β_3	α_4	β_3	γ_2	δ_1	δ_2	δ_3
ε_2	ε_1	δ_2	γ_3	β_4	α_5	β_4	γ_3	δ_2	ε_1	ε_2
ζ_1	ε_2	δ_3	γ_4	β_5	α_6	β_5	γ_4	δ_3	ε_2	ζ_1

Fig. 16.

9. THE TOTAL CORRECTION FOR 3 LAYERS AND 5 LAYERS OF WIRES.

We have already found the correction for the first layer of 8 wires and the second layer of 16 wires. The correction for the third layer of 24 wires may now easily be calculated. It is

$$\begin{aligned} \varDelta M_3 - \varDelta M_2 &= 4\,(\alpha_4 + 2\,\beta_3 + 2\gamma_2 + \delta_1) \qquad (29) \\ &= 4\,(102.5 + 47.2 - 69.6 - 25.6) \times 10^{-6} \\ &= 0.000218 \\ \therefore\ \varDelta M_3 &= 0.017708 + 000218 \\ &= 0.017926 \end{aligned}$$

$\varDelta M_3$ is the correction for 3 layers of 48 wires.

Fig. 16 gives the correction terms for each of 120 squares on the central square A, in a larger square of 11 x 11 unit squares. The four quadrants are symmetrical, and in each quadrant values are symmetrical about the diagonal.

1170	1589	1614	1619	1621	1622	1622		184	184	185	187	192	217	636
1589	1777	1793	1800	1804	1804.5			1.4	1.5	2	6	13	29	217
1614	1793	1796	1800	1803	1804.8			1.2	1.2	3	6	10	13	192
1619	1800	1800	1801	1803	1804.7				1.3	3	5	6	6	187
1621	1804	1803	1803	1804	1805				1	2	3	3	2	185
1622	1804.5	1804.8	1804.7	1805	1806				0	1	1.3	1.2	1.5	184
1622	1804.5	1804.8	1804.7	1805	1806				0	1	1.3	1.2	1.5	184
1621	1804	1803	1803	1804	1805				1	2	3	3	2	185
1619	1800	1800	1801	1803	1804.7				1.3	3	5	6	6	187
1614	1793	1796	1800	1803	1804.8			1.2	1.2	3	6	10	13	192
1589	1777	1793	1800	1804	1804.5			1.4	1.5	2	6	13	29	217
1170	1589	1614	1619	1621	1622	1622		184	184	185	187	192	217	636

Fig. 17.

To find the correction for 120 wires constituting five layers we find the logarithm of the g. m. d. of the square A from the space outside of it contained within the larger square, the latter consisting of 120 small squares. Then find the g. m. d. of the circle inscribed in the square A from the 120 circles inscribed in the 120 squares. The result is as follows:

$$\log R_{120} = 1.35674951, \text{ for the square}$$
$$\log R'_{120} = 1.35659900, \text{ for the circles}$$
$$\text{Difference} = .00015051$$

Multiplying by 120, $\Delta M_5 = .018061$ = total correction for 120 squares.

SUMMARY OF CORRECTIONS.

For 1st layer of	$8 = .016906$ ∴	$\Delta M_1 = .016906$ for 1 layer.
For 2d layer of	$16 = .000802$	$\Delta M_2 = .017708$ for 2 layers.
For 3d layer of	$24 = .000218$	$\Delta M_3 = .017926$ for 3 layers.
For 4th & 5th layers of	$72 = .000135$	$\Delta M_5 = .018061$ for 5 layers.

These are the values of the correction for mutual induction for a single turn of wire in a coil surrounded by other wires, assuming the wires are of small section and the curvature small. Such wires as lie on or near the surface of the coil will evidently have smaller values for ΔM.

CORRECTION FOR MUTUAL INDUCTANCE NEAR SURFACE OF A COIL.

To find the correction to be made for a particular wire in a coil of rectangular section add together the corrections due to the neighboring wires. Thus, let NOP be a portion of the boundary of the section of a coil, one turn of wire occupying each square. Let us find the correction for the corner wire A_1. The space to the right of N_1O_1P is a complete quadrant, and the correction due to it is one-fourth of .018061, or .004515. The wires in the column above A_1 give corrections a_2, a_3, a_4, etc. $= \Sigma a = .007188$. The sum of these is 0.01170 which is the total correction for mutual inductance for the corner wire. The corrections for B_1, Fig. 19, are as follows:

For quadrant $N_1O_1P_1$, $0.018061 \div 4 =$		.004515
For C_1, D_1, E_1, etc.	$= \Sigma a =$	.007188
For A_1,	$a_2 =$	.006528
	Sum $=$	.018231
For A_2, A_3, A_4, etc.,	$\Sigma \beta =$	$-.002343$
Total correction		.015888

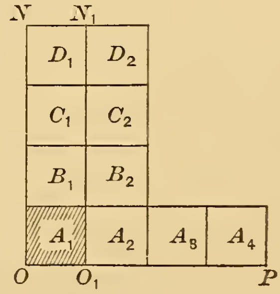

Fig. 18

In a similar manner the correction can be found for B_2, Fig. 20, or for any wire in a coil.

In Fig. 17 the values of ΔM are plotted on the left for the different wires, and the differences between the respective values and .01806, the maximum value, are plotted on the right, the numbers on the right being parts in 100,000. The differences are practically confined to the outer layer and a few wires in each corner. The average value of the differences for a coil of 10 x 10 turns is .00093, making the average correction $+.01806-.00093=+.01713$.

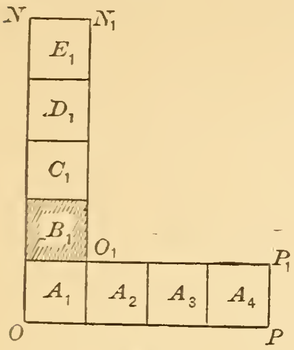

Fig. 19

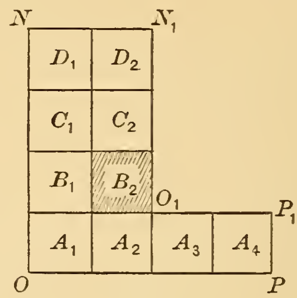

Fig. 20

$$\therefore \Delta M = 4\pi n a \times .01713. \tag{30}$$

10. INDEPENDENT DETERMINATION OF GEOMETRIC MEAN DISTANCE OF ADJACENT SQUARES.

The geometric mean distance of a square from itself is given by the expression:

$$\log R = \log a + \frac{1}{3}\log 2 + \frac{\pi}{3} - \frac{25}{12} \tag{31}$$

$$\therefore R = .447049 \times a$$

where R is the geometric mean distance and a is the side of the square.

The geometric mean distance of a rectangle from itself is [10]

$$\log R = \log\sqrt{a^2+b^2} - \frac{1}{6}\frac{a^2}{b^2}\log\sqrt{1+\frac{b^2}{a^2}} - \frac{1}{6}\frac{b^2}{a^2}\log\sqrt{1+\frac{a^2}{b^2}}$$

$$+\frac{2}{3}\frac{a}{b}\tan^{-1}\frac{b}{a} + \frac{2}{3}\frac{b}{a}\tan^{-1}\frac{a}{b} - \frac{25}{12} \tag{32}$$

where a and b are the breadth and length of the rectangle respectively.

When $b=2a$,

$$R = 0.670803 \times a$$

[10] Maxwell, II, § 692.

Suppose ABCD, Fig. 21, represents a rectangle in which the length is two and the breadth one, and EF divides it into two unit squares. Let R be the g. m. d. of the entire rectangle from itself, R_1 the g. m. d. of the square from itself, and R_2 the g. m. d. of one square from the other. Then $R=.670803$, $R_1=.447049$, and R_2 is to be determined.

By the principle of the geometric mean distance

$$2\log R=\log R_1+\log R_2 \qquad (33)$$
$$\text{or } R^2=R_1\,R_2$$
$$2\log_{10} R=\bar{1}.6531900$$
$$\log_{10} R_1=\bar{1}.6503551$$
$$\therefore \log_{10} R_2=0.0028349$$
$$R_2=1.006549$$
$$\log_e R_2=\ .006528=a_2$$

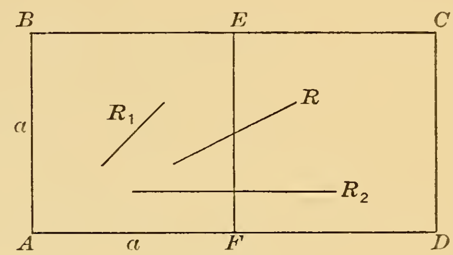

Fig. 21

This is the value of the geometric mean distance of adjacent squares already found by means of formula (8a), differing by less than one part in a million, this difference occurring because the logarithms are not carried out as far in this case as before.

11. SQUARES CORNER TO CORNER.

Let ABCD be a square of length 2 units, divided into four unit squares. Let R be the g. m. d. of the large square from itself $=.894098$; R_1 be the g. m. d. of each small square from itself $=.447049$; R_2 the g. m. d. of any small square from an adjacent one$=1.006549$ (see above) and R_3 the g. m. d. of two squares corner to corner, as 1 from 4 or 2 from 3. R_3 is to be determined. By the principle of the geometrical mean distance,

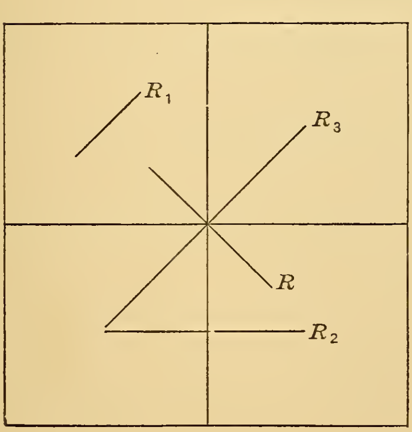

Fig. 22

$$4\log R=\log R_1+2\log R_2+\log R_3 \qquad (35)$$
$$\text{or, } R^4=R_1\,R_2^2\,R_3$$
$$\log_{10} R_1=\bar{1}.6503551$$
$$2\log_{10} R_2=0.0056698$$
$$\text{Sum}=\bar{1}.6560249$$
$$4\log R=\bar{1}.8055404$$
$$\log R_3=0.1495155$$
$$\tfrac{1}{2}\log 2=0.1505150$$
$$\therefore \log \frac{R_3}{\sqrt{2}}=\bar{1}.9990005$$
$$\therefore R_3=0.997701\times\sqrt{2}$$

That is, the g. m. d. of two squares, corner to corner, is 0.997701 times the distance between their centers. This is the result previously found from equation (17).

12. SQUARES NOT ADJACENT; $R\alpha_3$.

Let the rectangle of Fig. 23 consist of three unit squares. Let R = g. m. d. of the rectangle from itself, which formula (32) shows to be 0.894657; R_1 = g. m. d. of a single square from itself = 0.447049, R_2 = g. m. d. of one square on adjacent one = 1.006549, R_3 = g. m. d. of square 1 on square 3, which is to be determined. By the principle of the geometrical mean distance

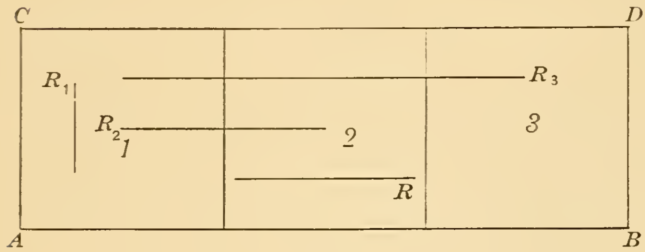

Fig. 23

$$9 \log R = 3 \log R_1 + 4 \log R_2 + 2 \log R_3 \tag{36}$$

$$\text{or, } R^9 = R_1^3\, R_2^4\, R_3^2$$

$$\text{and } R_3 = \sqrt{\frac{R^9}{R_1^3\, R_2^4}} = 2.001022$$

$$= 1.000511 \times 2a$$

for squares of side a.

$$\therefore \frac{R_3}{R_3{}'} = 1.000511 \text{ agreeing closely with the value found p. 9.}$$

$$\therefore a_3 = .000511$$

We have thus derived from the comparatively simple formulæ (31 and 32) the same values of a_2 and a_3 that were obtained from equations (8*a*) and (8).

13. SELF-INDUCTANCE OF A CIRCLE OF SQUARE SECTION.

THREE FORMULÆ COMPARED.

Problem 1.—Circle of square section.

As a first illustration of the use of the geometric mean distance in calculating inductances let us take the case of a single circular turn of wire of square section. The formulæ are more exact as the ratio of the radius to the side of the cross section is greater; hence, we may take $a=25$ cm and $b=c=0.1$ cm as a case favorable to high accuracy. The three formulæ to be compared will be Weinstein's, Stefan's, and Maxwell's, in the latter using the geometric mean distance of the square section. The formulæ are as follows for the case of a square section:

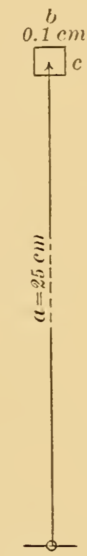

Fig. 24

Weinstein's:

$$L=4\pi a\left\{\log\frac{8a}{b}\cdot\left(1+\frac{b^2}{24a^2}\right)+.03657\,\frac{b^2}{a^2}-1.194913\right\} \tag{37}$$

Stefan's:

$$L=4\pi a\left\{\log\frac{8a}{b\sqrt{2}}\cdot\left(1+\frac{b^2}{24a^2}\right)-y_1+\frac{b^2}{16a^2}y_2\right\} \tag{38}$$

where $y_1=0.848340$ and $y_2=0.8162$ for a square.

Maxwell's:

$$L=4\pi a\left\{\log\frac{8a}{R}\cdot\left(1+\frac{3R^2}{16a^2}\right)-2-\frac{R^2}{16a^2}\right\} \tag{39}$$

where R is the geometric mean distance of the square section from itself $=.0447049$ cm in this case.

Substituting $a=25$ cm and $b=0.1$ cm we find the following values for the self-inductance:

By Weinstein's:

$$\frac{8a}{b}=2000. \qquad \log_e 2000=7.600902$$

$$\frac{1}{24}\frac{b^2}{a^2}\cdot\log\frac{8a}{b}=\ \ .000005$$

$$.03657\,\frac{b^2}{a^2}=\ \ .000001$$

$$\overline{7.600908}$$

$$-1.194913$$

$$\overline{6.405995}$$

$$4a=100\ \therefore\ \frac{L}{\pi}=640.5995 \text{ cm.}$$

By Stefan's:

$$\begin{aligned}
\log_e 2000 &= 7.600902 \\
-\tfrac{1}{2}\log 2 &= -0.346573 \\
& \overline{7.254329} \\
\frac{b^2}{24a^2}\log\frac{8a}{b\sqrt{2}} &= .000005 \\
\frac{b^2}{16a^2}y_2 &= .000001 \\
& \overline{7.254335} \\
-y_1 &= -0.848340 \\
& \overline{6.405995} \\
\therefore \frac{L}{\pi} &= 640.5995
\end{aligned}$$

By Maxwell's:

$$\frac{8a}{R} = \frac{200}{.0447049} = \frac{2000}{.447049}$$

$$\begin{aligned}
\log_e 2000 &= 7.600902 \\
-\log .447049 &= +.805087 \\
& \overline{8.405989} \\
\frac{3R^2}{16a^2}\times\log\frac{8a}{R} &= 0.000005 \\
& \overline{8.405994} \\
-\left(2+\frac{R^2}{16a^2}\right) &= -2.000000 \\
& \overline{6.405994}
\end{aligned}$$

$$\therefore \frac{L}{\pi} = 640.5994$$

These three values of L are practically identical. Stefan's formula is derived from Weinstein's, but Maxwell's is an independent one. This confirms the value of R for the square.

Problem 2.—Circle of Larger Section.

As already stated, the geometric mean distance is calculated on the basis of a straight conductor, which is equivalent to a circle of infinite radius. We see by the above example that it must be substantially the same for a circle where the radius is 250 times the side of the square as it is for a circle of infinite radius. For a square of 1.0 cm side, the radius of the circle being 25 cm, and hence the radius only 25 times the side of the square, the value of R would not be as exact, and hence the agreement among the three formulæ would not be as good as the foregoing. Using the above formulæ for this case, Fig. 25, we find the following results:

By Weinstein's and Stefan's formulæ, $L=410.3816$
By Maxwell's, using the g. m. d. . . $L=410.3750$
Difference $=$ .0066

The difference is one part in 60,000; the second value being smaller shows the g. m. d. is a little too great when applied to a circle where the curvature is considerable.

Problem 3.—Coil of Two Turns.

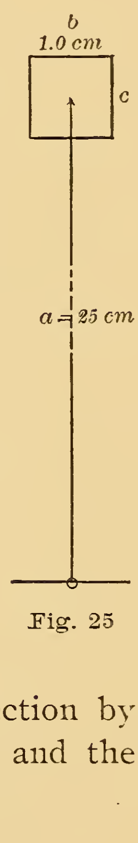

Fig. 25

Calculating the self-inductance of a coil of two turns of round wire by the method of summation, and also by Stefan's formula, we can obtain the correction for mutual inductance independently of the formulæ for geometric mean distance. The self-inductance of the coil of two turns is

$$L=2L_1+2M_{12}$$

Suppose $a=99.85$ cm and the diameter of the section of the wire is 0.1 cm, Fig. 26.

By Wien's formula $L_1=791.694990\times 4\pi$
By Maxwell's formula $M_{12}=697.521875\times 4\pi$

$$\therefore \frac{L_\sigma}{4\pi}=2978.43373 \text{ cm.}$$

The self-inductance of a coil of two turns of *square* section by Stefan's formula, the section of the coil being 1×2 mm and the radius the same as before is

$$\frac{L_u}{4\pi}=2949.55944$$

The correction to reduce to a round section is

$$na\,(.1380605),\ na=199.7,\ \therefore \frac{\Delta_1 L}{4\pi}=\quad 27.57068$$

$$\therefore L\div 4\pi=\text{sum}=2977.13012$$

Difference from $L_\sigma\div 4\pi=\quad 1.30361$

If there were no correction for mutual induction these two values should agree. Their difference is the correction in question. That is

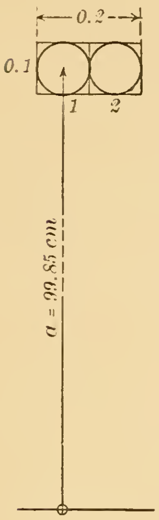

Fig. 26

$$\Delta_2 L = naa_2 = 1.30361$$

$$na = 199.7 \quad \therefore \quad a_2 = \frac{1.30361}{199.7} = .0065278$$

This is almost identically the value found for a_2 by the method of geometrical mean distances. If we had used .0065285 and applied the correction naa_2 based upon it we should have found $\Delta_2 L/4\pi = 1.30374$ and $L/4\pi = 2978.43386$. This differs from the value $L_\sigma/4\pi$ by the method of summation given above by less than one in 20,000,000. This is a very interesting confirmation of the result obtained by the method of geometrical mean distances, and shows that when the radius of the coil is large in comparison with the dimensions of the cross section that the formulæ are very accurate.

Problem 4.—Coil of Four Turns.

We will now find β_1, the correction for diagonal squares, by the method just employed to find a_2. The self-inductance of this coil by the method of summation is

$$L = 2L_1 + 2L_3 + 2M_{12} + 2M_{34} + 4M_{13} + 4M_{14} \qquad (40)$$

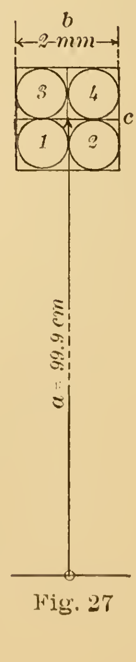

Fig. 27

As before, L_1 and L_3 are calculated by Wien's formula for circles and the Ms by Maxwell's formula for mutual inductances. The results are as follows:

$$\begin{aligned} 2L_1 &= 1583.3900 \\ 2L_2 &= 1585.1758 \\ 2M_{12} &= 1395.0438 \\ 2M_{34} &= 1396.6410 \\ 4M_{13} &= 2791.6843 \\ 4M_{14} &= 2653.1941 \\ \therefore L_\sigma/4\pi &= 11405.1290 \end{aligned}$$

By Stefan's and Weinstein's formula $L_u/4\pi$ is found on the assumption of a uniform distribution of current over the cross section of the coil.

$$\frac{L_u}{4\pi}=11345.6632$$

Correction to reduce from square to round wire $=na(.1380605)=399.6\times.1380605=\quad 55.1690$

$$\text{Sum}=\frac{L'}{4\pi}=11400.8322$$

$$\therefore \frac{\Delta_2 L}{4\pi}=\frac{L_\sigma-L'}{4\pi}=4.2968=\text{the correction for mutual induction.}$$

$$=na(2\alpha_2+\beta_1)$$

$$\therefore 2\alpha_2+\beta_1=\frac{4.2968}{399.6}=\quad .010753$$

$$2\alpha_2 \quad = \quad .013057$$

$$\therefore \beta_1 \quad =-.002304$$

By the method of geometric mean distances we found β_1 to be $-.002302$. This slight difference amounts to only one part in 15,000,000 of the whole value of the self-inductance of the coil. Thus we see that the two largest correction terms α_2 and β_1 derived by means of formulæ of self and mutual inductance agree with the values found by the method of geometric mean distances.

It is probable that Stefan derived his value for the correction in this way. As we have already seen, the effect of the eight nearest wires on any given wire, Fig. 28, is

$$4\alpha_2+4\beta_1=.026114-.009204$$
$$=.01691$$

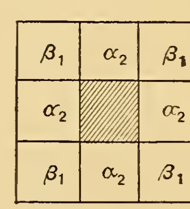

Fig. 28

and this is practically the value of E (page 4), which results from the correction given by Stefan. This neglects all the other wires and assumes the effect equal on all the wires of the section. This is indeed a very good first approximation to the correction, and amply accurate for most purposes.

Problem 5.—Coil of Four Turns in one Layer.

The self-inductance of a coil of four turns of wire in one layer by the method of summation is

$$L = 4L_1 + 6M_{12} + 4M_{13} + 2M_{14} \qquad (41)$$

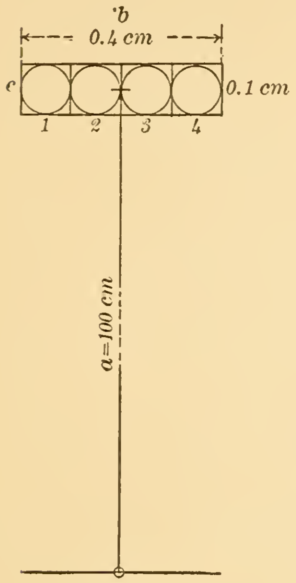

Fig. 29

The Ls and Ms computed as in the last article are as follows:

$$\begin{aligned} 4\,L_1 &= 12688.549\pi \\ 6\,M_{12} &= 16769.275\pi \\ 4\,M_{13} &= 10070.481\pi \\ 2\,M_{14} &= 4710.869\pi \\ \therefore L_\sigma &= 44239.174\pi \end{aligned}$$

By Stefan's formula, the self-inductance of a coil of four turns of square wire filling the entire section is, for the above dimensions:

$$L_u = 44001.729\pi$$

We must add to this the correction to reduce to a round section and the correction for mutual induction. The latter involves a_2, a_3, a_4. Taking each wire separately it will be seen that the sum of these corrections is $6a_2 + 4a_3 + 2a_4$

$$\begin{aligned} 6a_2 &= .0065285 \times 6 = .0391710 \\ 4a_3 &= .0005104 \times 4 = .0020416 \\ 2a_4 &= .0001025 \times 2 = .0002050 \\ &\qquad\qquad\qquad\quad .0414176 \\ E &= \text{average for each wire} = .0103544 \\ \text{Correction } C &= .1380605 \\ C + E &= .148415 \\ 4\pi a n &= 1600\pi \\ \therefore \Delta L &= .148415 \times 1600\pi = 237.464\pi \\ L_u &= 44001.729\pi \\ \therefore L &= 44239.193\pi \\ \text{Difference from } L_\sigma &= .019\pi \end{aligned}$$

This difference is less than one part in two million. Without the correction E for mutual induction (depending on a_2, a_3, a_4) the discrepancy would be 1,000 times as large.

Problem 6.—Coil of Ten Turns.

Let us take a coil of ten turns of round covered wire, the radius being 25 cm, diameter of bare wire being 0.8 mm, of covered wire 1.0 mm, the whole length of coil being 1 cm. The self-inductance of this coil has been calculated in a previous paper[11] to be 47385.82π cm using the method of summation.

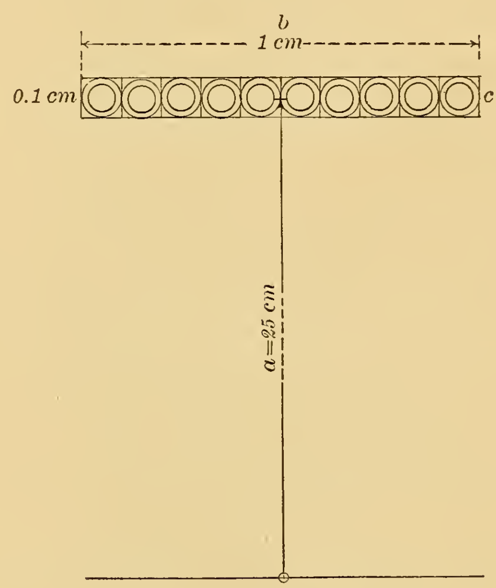

Fig. 30

By Stefan's formula (substituting $a=25$, $n=10$ $b=1.0$, $c=0.1$) we find

$$L_u=47011.93\pi \text{ cm}.$$

This assumes a uniform distribution of current over the rectangular section, as though the coil were wound with square wire having insulation of infinitesimal thickness. The correction to apply to reduce it to the actual winding of round insulated wire is as follows:

$$\Delta L=4\pi an\left[\log_e\frac{D}{d}+.1380605+\frac{1}{n}\Sigma a\right]$$

[11] Calculation of Self-Inductances of Single Layer Coils, Bulletin of Bureau of Standards, 2, p. 165; 1906.

Σa for a single layer coil of n turns is as follows:

$$\Sigma a = 2\left[(n-1)a_2 + (n-2)a_3 + (n-3)a_4 + (n-4)a_5 + (n-5)a_6 + \ldots\right] \quad (42)$$

For a coil of ten turns we have

$$
\begin{aligned}
.006528 \times 9 &= .058756 \\
.000511 \times 8 &= .004083 \\
.000102 \times 7 &= .000717 \\
.000033 \times 6 &= .000195 \\
.000013 \times 5 &= \underline{.000068} \\
\text{Sum } &= .063819 \\
\therefore\ \Sigma a &= \underline{.127638} \\
E = \frac{1}{n}\Sigma a &= .012764 \\
C &= .138060 \\
F = \log_e \frac{D}{d} &= \underline{.223144} \\
\text{Sum } &= .37397 \\
\Delta L &= 4\pi a n \times 0.37397 \\
\text{or } \Delta L &= \quad 383.97\pi \text{ cm} \\
L_u &= \underline{47011.93}\pi \text{ cm} \\
\therefore\ L &= 47385.90\pi \text{ cm} \\
\text{whereas } L\sigma &= \underline{47385.82}\pi \text{ cm} \quad \text{(See p. 31.)} \\
\text{Difference } &= \qquad .08\pi \text{ cm}
\end{aligned}
$$

That is, the difference between the self-inductance by the very accurate formulæ employed in the method of summation, and the value by Stefan's formula for a uniform distribution of current plus the corrections to reduce to the actual winding amounts to one part in 600,000.

Problem 7.—Coil of Twenty Turns.

$a = 25 \qquad b = 2$ cm $\qquad c = 0.1$ cm $\qquad n = 20$

Diameter of bare wire 0.6 mm, of covered wire 1.0 mm.

In the last case we obtained the self-inductance of the coil by two distinct methods, the first being the method of summation, the second by assuming the current uniformly distributed over the section and then applying the three corrections C, F, E. In this problem we may calculate L, first, by use of the current sheet formula and apply the corrections A and B to reduce to round wires, and then, second, by Stefan's formula for uniform distribution and apply the three corrections C, F, E and reduce to round wires.

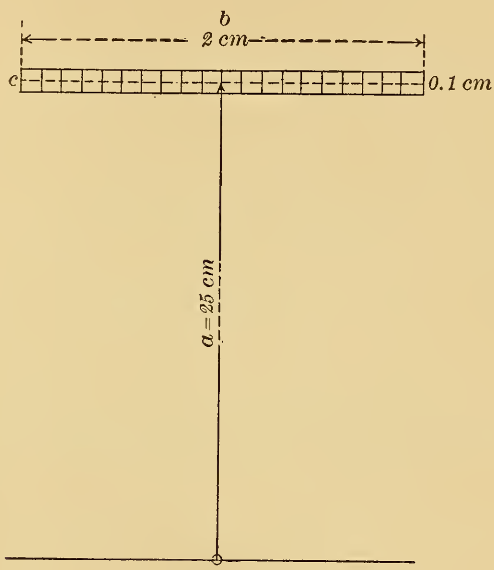

Fig. 31

Rayleigh's formula for a narrow current sheet is as follows:

$$L=4\pi an^2\left\{\log\frac{8a}{b}-0.5+\frac{b^2}{32a^2}\left(\log\frac{8a}{b}+\frac{1}{4}\right)\right\}$$

Here $\quad \frac{8a}{b}=100 \qquad \log_e 100=4.605170$

$$\frac{4}{20{,}000}\left(\log\frac{8a}{b}+\frac{1}{4}\right)=\underline{.000971}$$

$$4.606141$$
$$\underline{-0.500000}$$
$$4.106141$$

$$4\pi an^2=40{,}000\pi \therefore L_s=164{,}245.64\pi \text{ cm}$$

This is the self-inductance of a winding of 20 turns of infinitely thin tape, each turn being 1 mm wide, with edges touching without making electrical contact, which arrangement fulfills the conditions of a current sheet. To reduce this to the case of round wires we must apply the corrections A and B for self and mutual induction.[12]

$$\text{By Table VII, for } \frac{d}{D}=0.6, \quad A= \quad .0460$$
$$\text{By Table VIII, for } n=20, \quad B= \quad .2974$$
$$A+B= \quad .3434$$

$$4\pi an = \quad 2000\pi$$
$$\therefore \Delta L=4\pi an\,(A+B)= \quad 686.8\pi \text{ cm}$$
$$\therefore L=L_s-\Delta L=163{,}558.84\pi \text{ cm}$$

By Stefan's formula we find, substituting the above values of a, n, b, c, and taking $y_1=.548990$ and $y_2=.1269$

$$L_u=162{,}234.60\pi \text{ cm}$$

The correction $E=\frac{1}{n}\Sigma a$ is found by substituting 20 for n in equation (42). This gives $E=.01357$. The three corrections are then as follows:

$$C=.13806$$
$$F=.51082$$
$$E=.01357$$
$$.66245$$
$$\therefore \Delta L=4\pi an(C+F+E)=1324.90\pi \text{ cm}$$
$$\therefore L=L_u+\Delta L=163{,}559.50\pi \text{ cm}$$

This value of L is greater than the value found by the other method by only four parts in a million. Thus we see that the method of calculating L_u by Stefan's or Weinstein's formula and applying the corrections C, F, E gives practically identical results with the method of summation and also with the current sheet method *for short coils.* When, however, the coils are longer the agreement is not so good, for the reason that the formula of Weinstein (and Stefan's derived from it) are not as accurate when the section of the coil is greater. Thus if the coil in the above problem had been 5 cm long and 2.5 mm deep, and wound with 20 turns of heavier wire the difference would have been 1 part in 25,000 (still very good agreement), and if it were

[12] Rosa, Bulletin of Bureau of Standards, 2, p. 161; 1906.

10 cm long and 0.5 cm deep (the radius being 25 cm) it would have been 1 part in 2,200. For most experimental work, therefore, the formula is amply accurate.

Problem 8.—Coil of Sixteen Turns.

$$a=100. \quad b=c=4 \text{ mm.}$$

The coil is wound with round 1 mm wire. It is a somewhat tedious operation to calculate the self-inductance of a coil of 16 turns by the method of summation, but it seemed worth while to test the correction terms α, β, γ, δ, in this way. By taking a large radius and a small section the work is diminished without sacrificing the accuracy. The result is

$$L_\sigma=656{,}954.28\pi \text{ cm}$$

By Weinstein's formula, substituting $a=100$, $b=0.4$, $n=16$,

$$L_u=655{,}973.87\pi \text{ cm}$$

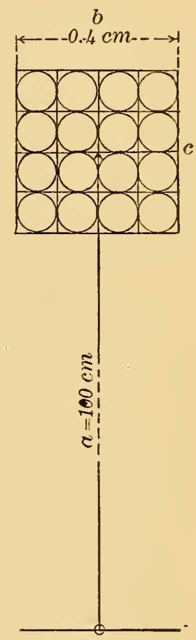

Fig. 32

To find the correction for mutual induction we have to consider the effect on each wire of the sixteen of all the other wires, and take the mean.

The correction α_2 applies twice to each of the four corner wires, three times to each of the 8 wires numbered 2, 3, 5, 9, 14, 15, 8, 12; and four times to each of the four wires 6, 7, 10, 11, Fig. 33. Thus

$$(2\times 4+3\times 8+4\times 4)\div 16=3$$

That is, α_2 applies three times to each wire *on the average.* In the same way we find that α_3 applies twice to each wire on the average and α_4 once. Considering all the correction terms in this way we find

$$\frac{1}{n}\Sigma(\alpha, \beta, \gamma, \delta)=E=3\alpha_2+2\alpha_3+\alpha_4+\frac{3}{2}\beta_2+\frac{9}{4}\beta_1+3\beta_2+\gamma_1+\gamma_2+\frac{\delta_1}{4} \qquad (43)$$

The sum of the coefficients is $15 = n - 1$

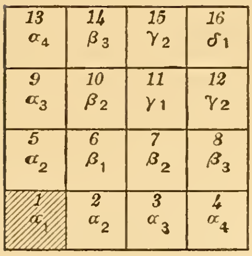

Fig. 33

$$
\begin{aligned}
3a_2 &= .019586 \\
2a_3 &= .001021 \\
a_4 &= .000102 \\
\frac{3}{2}\beta_3 &= \underline{.000035} = .020744 \\
\frac{9}{4}\beta_1 &= -.005179 \\
3\beta_2 &= -.000269 \\
\gamma_1 + \gamma_2 &= -.000166 \\
\frac{\delta_1}{4} &= -.000006 = \underline{-.005620} \\
&\therefore E = +.015124
\end{aligned}
$$

The correction F is zero, since we have assumed the diameter of the bare wire to be 1 mm, and hence $D = d$.

$$
\begin{aligned}
C &= .13806 \\
E &= \underline{.01512} \\
\text{Sum} &= .15318 \\
4\pi na &= 6400\pi \\
\varDelta L = 4\pi na(C+E) &= 980.35\pi \text{ cm}
\end{aligned}
$$

From above, $$L_u = \underline{655{,}973.87\pi} \text{ cm}$$

$$\therefore L = L_u + \varDelta L = 656{,}954.22\pi \text{ cm}$$

This value of L agrees with the value L_σ obtained by the method of summation within one part in 10,000,000. In the calculations of L_σ and L_u the values were carried out two more decimal places than are given above, so that the close agreement is not merely accidental. With the large radius and small section chosen the formulæ are very exact, and this gives an excellent test of the corrections, a, β, γ, δ, for mutual induction.

The relative importance of the corrections C, F, and E decreases with the number of turns, for $\varDelta L$ is proportional to n whereas L is proportional to n^2. Thus for a coil of a large number of turns it is quite unnecessary to know E accurately, so far as knowing the self-inductance for experimental purposes. If we take $C = .138$, $E = .018$, or $C + E = 0.156$, we shall be amply accurate, when n is large, for the most refined experimental work. For a coil of few turns E is smaller and relatively more important, and a value can be chosen from the above list that will be nearly right, or it can be calculated with great precision from the values of the corrections, α, β, γ, δ, etc., given in Fig. 16. These corrections of course are carried out much further than is necessary for any requirements of experimental purposes. Measurements can not be made of the section of the coil and of the diameter of the wire accurately enough to justify such refinements. It is better, however, that the formulæ should be more accurate than necessary rather than less so, and for the purpose of testing other formulæ it is desirable to be able to calculate self-inductance by this method with the highest precision. If further justification for these corrections is needed it must be found in the interest for its own sake of the subject of geometrical mean distance.

Summary of the values of E found for the various cases considered:

2 turns		$E=$	.006528	(Problem 3)
3 "	(one layer)	$E=$	.009045	
4 "	(two layers)	$E=$	.01691	(" 4)
4 "	(one layer)	$E=$	.01035	(" 5)
8 "	(two layers)	$E=$	.01335	
10 "	(one layer)	$E=$	.01276	(" 6)
20 "	(" ")	$E=$	.01357	(" 7)
16 "	(four layers)	$E=$	.01512	(" 8)
100 "	(ten layers)	$E=$	.01713	
400 "	(20×20)	$E=$	.01764	
1000 "	(50×20)	$E=$	.01778	
Infinite turns		$E=$	.01806	

14. CORRECTION FOR CURVATURE.

We have derived the α, β, γ, δ, ϵ corrections on the hypothesis of linear conductors, or of circular conductors in which the ratio of the radius to the side of the section is very great. When the section is increased the values of the equivalent distances between two coils is reduced; that is, the distance found by the method of geometrical mean distances is too great.

We can find the magnitude of this difference by computing the mutual inductance of two circular coils of square section by formulæ which I have given elsewhere,[13] and finding the distance apart of the equivalent circles. This distance is greater than the mean distance between centers of the coils when they are near and less when they are further apart, and it is always less than the geometrical mean distances for straight conductors when the radii are equal and the section considerable.

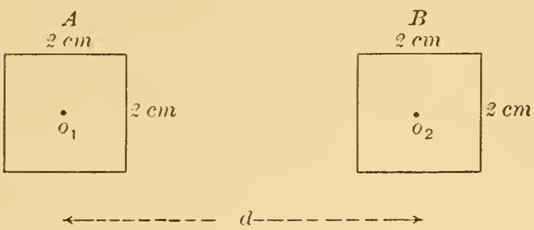

Fig. 34

The two straight conductors whose square sections are shown in Fig. 34 may be replaced by two single conductors O_1 O_2 near their centers, the slight displacement from the center being such as to be equivalent to the correction for section. The distance apart of O_1 O_2 would of course be the geo-

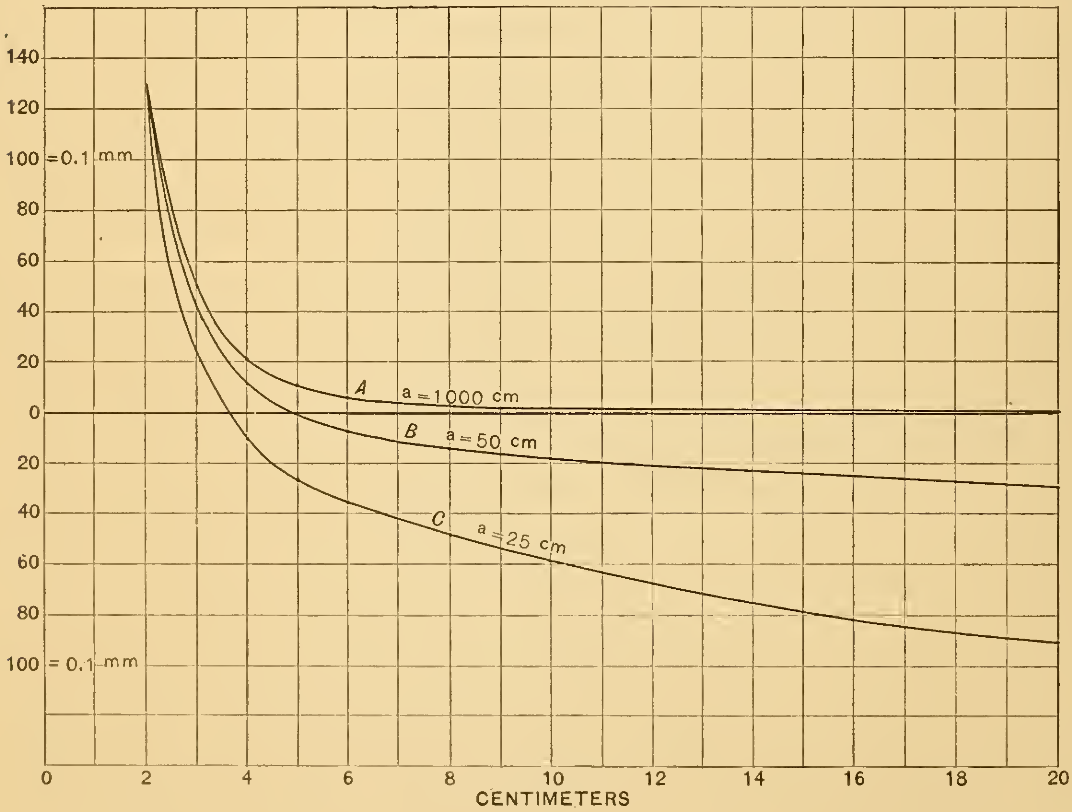

Fig. 35.

metrical mean distance of A and B. This is a little greater than the distance between centers, as we have seen for parallel squares in the position of A and B, Fig. 34. The curve A (Fig. 35) shows

[13] E. B. Rosa, Bulletin of Bureau of Standards, 2, p. 359; 1906.

how much the distance $O_1 O_2$ exceeds the distance between centers for distances between 2 cm (when A and B are in contact) and 20 cm. Each square represents 1 cm horizontally and .02 mm vertically. Curve A shows how rapidly the displacement Δd falls off as the conductors are further apart, Δd being only .001 mm at $d=$ 10 cm.

If, however, A B represent the sections of circular conductors the corrections by the method of geometrical mean distances as we have seen is only reliable when the section is very small in comparison with the radius. I have calculated the correction for section for the case of three circular coils, all of square section 2×2 cm, with radii 25, 50, and 1,000 cm; this correction is ΔM.[14] I have found the displacement Δd which gives the same value of ΔM by the following formula:

$$\Delta M = 4\pi a . \Delta d \left\{ \log \frac{8a}{d} \left(\frac{3d}{8a^2} - \frac{15d^3}{256a^4} + \frac{105d^5}{64.128a^6} \right) - \left(\frac{1}{d} + \frac{5d}{16a^2} - \frac{77d^3}{1024a^4} + \frac{282d^5}{(128)^2 a^6} \right) \right\} \tag{44}$$

These displacements are plotted in the curves A, B, C, Fig. 35. Curve A represents the coil of 1,000 cm radius, which is 500 times the side of the section, and approximates closely to the case of a straight conductor, having sensibly the same correction for section (represented by the displacement) as the straight conductors. The correction for section is negative (represented by a positive value for Δd) in all cases when the conductors are near each other, but quickly becomes positive for the circular conductors of radius 25 and 50 cm as the distance increases. In the case of the coils of smaller radius the negative displacement Δd is relatively great at the larger distances, although it is actually very small; the greatest value for $d=20$ for the smallest circle is less than 0.1 mm.

These curves show very forcibly how appreciable is the correction of the geometrical mean distance for curvature; or, in other words, that the geometrical mean distances calculated for straight conductors can be applied to circular conductors only when the distances between conductors is very small in comparison with the radius.

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[14] Bulletin of Bureau of Standards, 2, p. 349; 1906, equation (41).

APPENDIX.

Table of Inverse Tangents to nine decimal places.

$\tan^{-1}\frac{1}{12}$	0.083 141 232	$\tan^{-1}\frac{1}{3}$	0.321 750 554
$\tan^{-1}\frac{1}{11}$	.090 659 887	$\tan^{-1}\frac{1}{2}$	.463 647 609
$\tan^{-1}\frac{1}{10}$	.099 668 652	$\tan^{-1}\frac{2}{11}$	.179 853 506
$\tan^{-1}\frac{1}{9}$	.110 657 221	$\tan^{-1}\frac{2}{5}$	.380 506 377
$\tan^{-1}\frac{1}{8}$	.124 354 994	$\tan^{-1}\frac{2}{3}$	.588 002 603
$\tan^{-1}\frac{1}{7}$	.141 897 055	$\tan^{-1}\frac{3}{4}$	.643 501 109
$\tan^{-1}\frac{1}{6}$	.165 148 681	$\tan^{-1}\frac{3}{5}$	.540 419 500
$\tan^{-1}\frac{1}{5}$	.197 395 560	$\tan^{-1}\frac{4}{5}$	.674 740 942
$\tan^{-1}\frac{1}{4}$	.244 978 663	$\tan^{-1}\frac{5}{6}$	.694 738 276

$$\tan^{-1} 3 = \frac{\pi}{2} - \tan^{-1}\frac{1}{3}$$

$$\tan^{-1} 2 = \frac{\pi}{2} - \tan^{-1}\frac{1}{2} \text{ etc.}$$

Table of Constants for Stefan's Equation

b/c or c/b	y_1	y_2	b/c or c/b	y_1	y_2
0.00	0.50000	0.1250	0.55	0.80815	0.3437
0.05	.54899	.1269	0.60	.81823	.3839
0.10	.59243	.1325	0.65	.82648	.4274
0.15	.63102	.1418	0.70	.83311	.4739
0.20	.66520	.1548	0.75	.83831	.5234
0.25	.69532	.1714	0.80	.84225	.5760
0.30	.72172	.1916	0.85	.84509	.6317
0.35	.74469	.2152	0.90	.84697	.6902
0.40	.76454	.2423	0.95	.84801	.7518
0.45	.78154	.2728	1.00	.84834	.8162
0.50	.79600	.3066			

Table of Napierian Logarithms to nine decimal places for Numbers from 1 to 100.

1	0.000 000 000	51	3.931 825 633
2	0.693 147 181	52	3.951 243 719
3	1.098 612 289	53	3.970 291 914
4	1.386 294 361	54	3.988 984 047
5	1.609 437 912	55	4.007 333 185
6	1.791 759 469	56	4.025 351 691
7	1.945 910 149	57	4.043 051 268
8	2.079 441 542	58	4.060 443 011
9	2.197 224 577	59	4.077 537 444
10	2.302 585 093	60	4.094 344 562
11	2.397 895 273	61	4.110 873 864
12	2.484 906 650	62	4.127 134 385
13	2.564 949 357	63	4.143 134 726
14	2.639 057 330	64	4.158 883 083
15	2.708 050 201	65	4.174 387 270
16	2.772 588 722	66	4.189 654 742
17	2.833 213 344	67	4.204 692 619
18	2.890 371 758	68	4.219 507 705
19	2.944 438 979	69	4.234 106 505
20	2.995 732 274	70	4.248 495 242
21	3.044 522 438	71	4.262 679 877
22	3.091 042 453	72	4.276 666 119
23	3.135 494 216	73	4.290 459 441
24	3.178 053 830	74	4.304 065 093
25	3.218 875 825	75	4.317 488 114
26	3.258 096 538	76	4.330 733 340
27	3.295 836 866	77	4.343 805 422
28	3.332 204 510	78	4.356 708 827
29	3.367 295 830	79	4.369 447 852
30	3.401 197 382	80	4.382 026 635
31	3.433 987 204	81	4.394 339 155
32	3.465 735 903	82	4.406 719 247
33	3.496 507 561	83	4.418 840 608
34	3.526 360 525	84	4.430 816 799
35	3.555 348 061	85	4.442 651 256
36	3.583 518 938	86	4.454 347 296
37	3.610 917 913	87	4.465 908 119
38	3.637 586 160	88	4.477 336 814
39	3.663 561 646	89	4.488 636 370
40	3.688 879 454	90	4.499 809 670
41	3.713 572 067	91	4.510 859 507
42	3.737 669 618	92	4.521 788 577
43	3.761 200 116	93	4.532 599 493
44	3.784 189 634	94	4.543 294 782
45	3.806 662 490	95	4.553 876 892
46	3.828 641 396	96	4.564 348 191
47	3.850 147 602	97	4.574 710 979
48	3.871 201 011	98	4.584 967 479
49	3.891 820 298	99	4.595 119 850
50	3.912 023 005	100	4.605 170 186